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MORE ON MONADIC LOGIC
PART D: A NOTE ON ADDITION OF THEORIES

BY

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ABSTRACT

We improve somewhat some of the results from Baldwin and Shelah [BSh 156], closing a small gap therein (see [BSh 156], pages 248, lines 19 ff.; 255, second and third paragraphs; 256, following 3.2.10; and 257 following 3.2.12).

In [BSh 156], Baldwin and the author classified all theories of the form $(T, \mathcal{L})$ where $(T, \mathcal{L})$ is the collection of $\mathcal{L}$-sentences valid in models of the complete first-order theory T and $\mathcal{L}$ is one of the following: second-order logic, permutational logic, monadic logic. This classification necessitated in part the computation of bounds for certain kinds of Hanf numbers. For example, if $\alpha \geqq \omega$ and T is a countable $\aleph_1$-decomposable theory, then ([BSh 156] 3.2.9, p. 255)

$$(*) \qquad H^T_{L^\alpha_{\infty,\mu}(\mathrm{Mon})} \leqq (\beth_{1+\alpha+1}(\mu))^+,$$

where $H^T_{L^\alpha_{\infty,\mu}(\mathrm{Mon})}$ is the Hanf number of $L^\alpha_{\infty,\mu}(\mathrm{Mon})$ for theories relative to T. [In more detail, $H^T_{L^\alpha_{\infty,\mu}(\mathrm{Mon})}$ is the least cardinal κ such that for any $L^\alpha_{\infty,\mu}(\mathrm{Mon})$-theory Φ in the language of T, if $T \cup \Phi$ has a model of power κ, then $T \cup \Phi$ has models of arbitrarily large power. Again, see 3.2.9 in [BSh 156]: the logic $L^\alpha_{\infty,\mu}(\mathrm{Mon})$ is defined on page 245 (3.1.1(b)(iv)).]

Now in fact, as stated on pages 248, 255, 256 and 257 of [BSh 156], the author could improve the bound $(*)$, claiming the following theorem: *if* $\alpha \geqq \omega$ *and* T *is a countable*, $|T|^+$*-decomposable theory, then*

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$$(**) \qquad H^{T}_{L^{\alpha}_{\infty,\mu}(\mathrm{Mon})} \leqq (\beth_{1+\alpha}(\mu)^{|T|})^{+}.$$

Theorem 3 of this note presents a proof of (**) in the notation and framework of [BSh 156] and [ShA 1]. Briefly put, the basic strategy of [BSh 156] in bounding $H^{T}_{L^{\alpha}_{\infty,\mu}(\mathrm{Mon})}$ is to bound first the total number of possible $mT^{\alpha}_{\bar{k}}(M)$; then the assumption that T is $\aleph_1$-decomposable — so that M has a tree decomposition as a free union of small models — allows one to blow up M to a model of arbitrarily large power. In Definition 1 of this note, we shall formulate a principle $(*)(L, \alpha, \bar{k}, \lambda, \mu)$ which asserts the existence of a certain Boolean algebra B of power at most λ such that $|mT^{\alpha}_{\bar{k}}(L)| \leqq 2^{|B|}$ (see 3.1.1(a) in [BSh 156] for the definitions of $mT^{\alpha}_{\bar{k}}(M)$, $mT^{\alpha}_{\bar{k}}(L)$). It suffices then to prove in Theorem 3 the appropriate instances of $(*)(L, \alpha, \bar{k}, \lambda, \mu)$ in order to deduce the bound (**) for $H^{T}_{L^{\alpha}_{\infty,\mu}(\mathrm{Mon})}$.

Now let us provide the details relevant to Theorem 3.

1. Definition. Let $(*)(L, \alpha, \bar{k}, \lambda, \mu)$ be the statement:

(A) there is a Boolean algebra B of subsets of $mT^{\alpha}_{\bar{k}}(L)$, $\| B \| \leqq \lambda$ (this defines a topology, generated by the members of the Boolean algebra as the family of clopen sets) such that the following holds:

(B) *if* for $l = 0, 1$, $M^l = \Sigma_{i \in I_l} M^l_i$, $g^l : B \rightarrow$ cardinals is defined by $g^l(b) = |\{i \in I^l : mT^{\alpha}_{\bar{k}}(M^l_i) \in b\}|$ and

$$(\forall b \in B)[\mathrm{Min}\{g^0(b), \mu\} = \mathrm{Min}\{g^1(b), \mu\}],$$

then $mT^{\alpha}_{\bar{k}}(M^0) = mT^{\alpha}_{\bar{k}}(M^1)$.

[For the relevant definition, see [BSh 156] 3.1.8.]

2. Remark. Using (B) for I_0, I_1 singletons we get that any two members of $mT^{\alpha}_{\bar{k}}(L)$ are separated by some $b \in B$, hence $|mT^{\alpha}_{\bar{k}}(L)| \leqq 2^{|B|}$ and the topology which B indexes is Hausdorff.

3. Theorem. *For a sequence $\bar{k}$ of ordinals, a cardinal κ and an ordinal α, we define by induction on α (for all $\bar{k}$, L) the cardinals*

$$\lambda^{\alpha}_{\kappa,\bar{k}} \geqq \mu^{\alpha}_{\kappa,\bar{k}},$$

$$\alpha = 0, \qquad \lambda^{\alpha}_{\kappa,\bar{k}} = 2^{|L|} + \aleph_0, \qquad \mu^{\alpha}_{\kappa,\bar{k}} = \aleph_0,$$

$$\alpha + 1, \qquad \lambda^{\alpha+1}_{\kappa,\bar{k}} = 2^{\lambda^{\alpha}_{\kappa+|k(\alpha)|,\bar{k}}}, \qquad \mu^{\alpha+1}_{\kappa,\bar{k}} = (\lambda^{\alpha}_{\kappa+|k(\alpha)|,\bar{k}})^{+},$$

$$\alpha = \delta \text{ limit}, \qquad \lambda^{\alpha}_{\kappa,\bar{k}} = \sum_{\beta<\alpha} \lambda^{\beta}_{\kappa,\bar{k}}, \qquad \mu^{\alpha}_{\kappa,\bar{k}} = \sum_{\beta<\alpha} \mu^{\beta}_{\kappa,\bar{k}}.$$

Then $(*)(L, \alpha, \bar{k}, \lambda^{\alpha}_{|L|,\bar{k}}, \mu^{\alpha}_{|L|,\bar{k}})$.

PROOF. By induction on α. The case $\alpha = 0$ is Claim 5. The case α successor is Claim 4 and the case α limit is Claim 7.

4. LEMMA. *Suppose* $(*)(L + \bar{P}, \alpha, \bar{k}, \lambda, \mu)$ *holds and this is exemplified by the Boolean algebra* B, $\mu \leqq \lambda$ *and* $\bar{P} = \langle P_i : i < \bar{k}(\alpha)\rangle$. *Then* $(*)(L, \alpha + 1, \bar{k}, 2^{\lambda}, \lambda^{+})$ *holds.*

PROOF. Let $X = mT^{\alpha}_{\bar{k}}(L + \bar{P})$, $Y = mT^{\alpha+1}_{\bar{k}}(L)$. So B is a Boolean algebra of subsets of X, $\| B \| \leqq \lambda$ and, by 2, $|X| \leqq 2^{\lambda}$. We define a Boolean algebra C of subsets of Y. It is the closure by intersection of $\leqq \lambda$ many elements and by complements of the family of basic elements, where the basic elements are $\{t \in Y : s \in T\}$ (for $s \in X$) or $\{t \in Y : (\exists s \in S) s \in t\}$ where S is a closed or open subset of X.

We now prove that $(*)(L, \alpha + 1, \bar{k}, 2^{\lambda}, \lambda^{+})$ is exemplified by C.

First note that $|C| \leqq 2^{\lambda}$: the number of clopen subsets of X (by the topology which B induces) is exactly $\| B \| \leqq \lambda$, hence the number of open subsets is $\leqq 2^{\lambda}$, hence the number of closed subsets is $\leqq 2^{\lambda}$ and the closure under intersection of $\leqq \lambda$ and complementation does not change this.

Secondly, for $l = 0, 1$ let $M^l = \Sigma_{i \in I^l} M^l_i$, and let $g^l : C \to$ cardinals be defined by $g^l(c) = |\{i \in I^l : mT^{\alpha}_{\bar{k}}(M^l_i) \in b\}|$ and suppose that

$$(\forall c \in C)[\mathrm{Min}\{g^0(c), \lambda^{+}\} = \mathrm{Min}\{g^1(c), \lambda^{+}\}].$$

We shall prove that $mT^{\alpha+1}_{\bar{k}}(M^0) = mT^{\alpha+1}_{\bar{k}}(M^1)$.

By the symmetry it is enough to prove the following: we are given $\bar{P}^0_i$ $(i \in I^0)$ (a sequence of $k(\alpha)$ subsets of M^0_i); and we shall find $\bar{P}^1_i$ $(i \in I^1)$ such that

$$mT^{\alpha}_{\bar{k}}\left(\sum_{i \in I^0} (M^0_i \bar{P}^0_i)\right) = mT^{\alpha}_{\bar{k}}\left(\sum_{i \in I^1} (M^1_i, \bar{P}^1_i)\right).$$

Let $S = \bigcap \{b \in B$: there are $\leqq \lambda$ elements $i \in I^0$ such that $mT^{\alpha}_{\bar{k}}(M^0_i, \bar{P}^0_i)$ is not in $b\}$ (remember B is a family of subsets of X). Clearly S is a closed subset of X. Now

$$\{i \in I^0 : mT^{\alpha+1}_{\bar{k}}(M^0_i) \cap S = \varnothing\}$$

$$\subseteq \{i \in I^0 : mT_{\bar{k}}(M^0_i, \bar{P}^0_i) \notin S\}$$

$$\subseteq \bigcup_{b \in B} \{i \in I^0 : mT^{\alpha}_{\bar{k}}(M^0_i, \bar{P}^0_i) \in b \text{ and } (\exists^{\leqq\lambda} j \in I^0)[mT^{\alpha}_{\bar{k}}(M^0_j, \bar{p}^0_1) \in b]\}$$

which has power $\leqq \lambda$ (as $\| B \| \leqq \lambda$).

Let

$$A_0 = \{ i \in I^0 : mT_k^\alpha(M_i^0, \bar{P}_i^0) \notin S \} \qquad (\text{so } |A_0| \leqq \lambda).$$

Let $A_1 = \{ i \in I^0 : mT_k^{\alpha+1}(M_i^0) \cap S = \emptyset \}$ so $A_1 \subseteq A_0$. We can choose for each $i \in A_1$ a member c_i of C such that $mT_k^{\alpha+1}(M_i^0) \in c_i$, and $t \in c_i \Rightarrow t \cap S = \emptyset$ (remember that members of Y are subsets of X). If $i, j \in A_1$, $mT_k^{\alpha+1}(M_j^0) \neq mT_k^{\alpha+1}(M_i^0)$ *then* for some $x_{i,j} \in X$, $[x_{i,j} \in mT_k^{\alpha+1}(M_i^0) \Leftrightarrow x_{i,j} \notin mT_k^{\alpha+1}(M_1^0)]$. So we can replace c_i by

$$c_i^1 \overset{\text{def}}{=} \{ y \in Y : y \in c_i \text{ and if } x_{i,j} \text{ is defined } x_{i,j} \in y \Leftrightarrow x_{i,j} \in mT_k^{\alpha+1}(M_i^0) \}.$$

So if $i, j \in A_1$, $mT_k^{\alpha+1}(M_j^0) \neq mT_k^{\alpha+1}(M_i^0)$ implies $mT_k^{\alpha+1}(M_j^0) \notin c_i$ and even $c_j \cap c_i = 0$ (remember the definition of C). So for $c \leqq c_i$, $g^1(c) \leqq g^1(c_i) \leqq \lambda$ and $g^1(c) > 0$ iff $g^1(c) = g^1(c_i)$ iff $mT_k^{\alpha+1}(M_i^0) \in c$.

As we have assumed $(\forall c \in C)[\text{Min}(g^0(c), \lambda^+) = \text{Min}\{ g^1(c), \lambda^+ \}]$ the same holds for g^1, so we can find a one-to-one mapping h from A_1 onto

$$A_1^1 = \{ i \in I^1 : mT_k^{\alpha+1}(M_i^1) \cap S = \emptyset \}$$

such that

$$mT_k^{\alpha+1}(M_i^0) = mT_k^{\alpha+1}(M_{h(i)}^2).$$

Hence we can find $\bar{P}_j^1$ (for $j \in A_1^1$) such that

$$mT_k^\alpha(M_i^0, \bar{P}_i^0) = mT_k^\alpha(M_{h(i)}^1, \bar{P}_{h(i)}^1).$$

We (similarly to the above choice) can now define $c_i \in C$ for $i \in A_0 - A_1$ such that: $[i \in A_0 - A_1 \wedge j \in A_1 \wedge i \neq j \Rightarrow c_i \cap c_j = 0]$,

$$[mT_k^{\alpha+1}(M_i^0) \neq mT_k^{\alpha+1}(M_j^0) \wedge i \in A_1 - A_0 \Rightarrow c_i \cap c_j = 0]$$

and $(\forall i \in A_0 - A_1)(\forall t \in c_i)[t - S \neq \emptyset]$ ($t \in c_i$ implies $t \in Y$ hence $t \subseteq X$) and $(\forall t \in c_i)[mT_k^\alpha(M_i^0, \bar{P}_i^0) \in t]$. Now we can find a one-to-one function f from $A_0 - A_1$ into $I^1 - A_1^1$ such that $mT_k^{\alpha+1}(M_{f(i)}^1) \in c_i$ for $i \in A_0 - A_1$; then we can define $\bar{P}_{f(i)}^1$, such that $mT_k^\alpha(M_{f(i)}^1, \bar{P}_{f(i)}^1) = mT_k^\alpha(M_i^0, \bar{P}_i^0)$.

Now for every $b \in B$, $b \subseteq S$, we know that $E_b = \{ i \in I^0 : mT_k^\alpha(M_i^0, \bar{P}_i^0) \in b \}$ has power $\geqq \lambda^+$ and $|B| \leqq \lambda$, so it is well known that we can find $E_b' \subseteq E_b$, $|E_b'| = \lambda$, $E_b' \cap A_1 = \emptyset$, $E_{b_1}' \cap E_{b_2}' = \emptyset$ for $b_1 \neq b_2$. By the hypothesis on g_1 each

$$E_b^1 = \{ i \in I^1 : mT_k^{\alpha+1}(M_i^1) \cap b = \emptyset \}$$

has power $\geqq \lambda^+$ for $b \subseteq S (b \in B)$; so we can find a one-to-one mapping f_1 from $\bigcup \{ E_b' : b \in B, b \subseteq S \}$ to I^1-(Rang $f \cup$ Rang h) such that:

$$i \in E'_b \Rightarrow mT_{\bar{k}}^{\alpha+1}(M^1_{f_1(i)}) \cap b \neq \varnothing .$$

So we can define $\bar{P}^1_{f_1(i)}$ such that for $i \in E'_b$ where $b \in B$, $b \subseteq S$ such that $mT_{\bar{k}}^{\alpha}(M^1_i, \bar{P}^1_i) \in b$.

We define $\bar{P}^1_i$ for $i \in I^1 - \bigcup \mathrm{Rang}(f \cup f_1 \cup h)$ such that $mT_{\bar{k}}^{\alpha}(M^1_i, \bar{P}^1_i) \in S$ (not hard as $i \notin A^1_1$).

Now we apply the induction hypothesis.

5. Claim. $(*)(L, 0, \bar{k}; \lambda, \mu)$ holds with λ being the power of the Boolean algebra generated by the relevant $\exists \bar{x}\varphi$, φ conjunction of atomic and negation of atomic formulas, $\mu = \aleph_0$ (so for $k(-1) = \aleph_0$, L countable, $|mT_{\bar{k}}^0(L)| = 2^{2^{\aleph_0}}$, $|B| = 2^{\aleph_0}$).

6. Remark. Claim 5 raises the thought that it may be better to define $mT_{\bar{k}}^0(M)$ as $\{\exists \bar{x}\varphi : \varphi$ q.f. finite, $l(\bar{x}) < k(-2)\}$. So for L countable $|mT_{\bar{k}}^0(L)| = 2^{\aleph_0}$, $|B| = \aleph_0$ which seems more reasonable and I do not see any bad effect.

7. Claim. If δ is limit and $(*)(L, \alpha, \bar{k}, \lambda_\alpha, \mu_\alpha)$ is exemplified by B_α for $\alpha < \delta$, then $(*)(L, \delta, \bar{k}, \Sigma_\alpha \lambda_\alpha, \Sigma \mu_\alpha)$ is true (assuming $\Sigma_\alpha \lambda_\alpha$ is infinite, a triviality).

Proof. For $\alpha < \beta$ and L, let $\pi^{L,\bar{k}}_{\alpha,\beta}$ be the function from $mT_{\bar{k}}^{\beta}(L)$ to $mT_{\bar{k}}^{\alpha}(L)$ such that: if $x = mT_{\bar{k}}^{\beta}(M)$ then $\pi^{L,\bar{k}}_{\alpha,\beta}(x) = mT_{\bar{k}}^{\alpha}(M)$. Let B be the Boolean algebra of subsets of $mT_{\bar{k}}^{\delta}$ generated by

$$\{(\pi^{L,\bar{k}}_{\alpha,\delta})^{-1}(b) : \alpha < \delta, b \in B_\alpha\}$$

where B_α exemplifies $(*)(L, \alpha, \bar{k}, \lambda_\alpha, \mu_\alpha)$.

8. Discussion. (a) Is it worthwhile to make the general addition theory (i.e., I a structure) like what we do here?

(b) We can waive "B is a Boolean subalgebra"; for the finitary cases this saves us from meaninglessness (as B is necessarily the family of all subsets).

(c) We can also try to make $mT_{\bar{k}}^{\alpha}$ "grow" more slowly with α, e.g., in the case we look at partitions, we first take any coarser division with an *a priori* bounded number of parts. We shall still have addition theorems.

References

[ShA 1] S. Shelah, *Classification Theory and the Number of Non-isomorphic Models*, North-Holland Publ. Co., Amsterdam, 1978, 542 pp. +xvi.

[BSh 156] J. Baldwin and S. Shelah, *Classification of theories by second order quantifiers*, Proc. 1980/1 Jerusalem Model Theory Year, Notre Dame J. Formal Logic **26** (1985), 229–303.